# PERTURBATIONS OF BANACH FRAMES AND ATOMIC DECOMPOSITIONS

OLE CHRISTENSEN† AND CHRISTOPHER HEIL‡

**Abstract.** Banach frames and atomic decompositions are sequences which have basis-like properties but which need not be bases. In particular, they allow elements of a Banach space to be written as combinations of the frame or atomic decomposition elements in a stable manner. However, these representations need not be unique. Such flexibility is important in many applications. In this paper, we prove that frames and atomic decompositions in Banach spaces are stable under small perturbations. Our results are strongly related to classic results on perturbations of Paley/Wiener and Kato. We also consider duality properties for atomic decompositions, and discuss the consequences for Hilbert frames.

**Key words.** atomic decompositions, Banach frames, frames, perturbations

**AMS(MOS) subject classifications.** 42C99, 46B99, 46C99

**1. Introduction.** Frames for Hilbert spaces were introduced by Duffin and Schaeffer [DS] as part of their seminal research in nonharmonic Fourier series. Daubechies, Grossmann, and Meyer [DGM] later found a fundamental new application, to wavelet and windowed Fourier transforms. Frames continue to play an important role in each of these areas.

A set of vectors $\{y_i\}$ in a Hilbert space $H$ is a (*Hilbert*) *frame* if the norms $\|x\|_H$ and $\|\{\langle x, y_i\rangle\}\|_{\ell^2}$ are equivalent. Define $Ux = \{\langle x, y_i\rangle\}$. Then $U^*Ux = \sum \langle x, y_i\rangle\, y_i$ is an invertible mapping of $H$ onto itself. With $\tilde{y}_i = (U^*U)^{-1} y_i$, we have the *reconstruction formulas*

$$x \;=\; \sum \langle x, \tilde{y}_i\rangle\, y_i \;=\; \sum \langle x, y_i\rangle\, \tilde{y}_i. \tag{1}$$

The collection $\{\tilde{y}_i\}$ also forms a frame, the *dual frame* of $\{y_i\}$. The representations in (1) need not be unique: $\{y_i\}$ need not be a basis. A frame which is a basis must be a Riesz basis. Conversely, all Riesz bases are frames. The basic theory of frames in Hilbert spaces can be found in Duffin and Schaeffer's original paper [DS], Young's classic text [Y], Daubechies' paper [D1] and book [D2], or the research-tutorial [HW].

Frames were extended to Banach spaces by Gröchenig [G]. In Hilbert spaces, it is a remarkable fact that the norm equivalence hypothesis leads to the reconstruction for-

† Institut für Mathematik, Universität für Bodenkultur, Gregor Mendel–Strasse 33, A-1180 Wien, Austria (olechr@pap.univie.ac.at).

‡ School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160 USA and The MITRE Corporation, Bedford, Massachusetts 01730 USA (heil@math.gatech.edu). This author was partially supported by National Science Foundation Grant DMS-9401340 and the MITRE Sponsored Research Program.

mula (1). This does not hold in Banach spaces in general. A decomposition of a Banach space is therefore defined as follows.

Definition 1. Let $X$ be a Banach space and let $X_d$ be an associated Banach space of scalar-valued sequences indexed by $\mathbf{N} = \{1, 2, 3, \ldots\}$. Let $\{y_i\}_{i \in \mathbf{N}} \subset X'$ and $\{x_i\}_{i \in \mathbf{N}} \subset X$ be given. If:

(a) $\{\langle x, y_i \rangle\} \in X_d$ for each $x \in X$,

(b) the norms $\|x\|_X$ and $\|\{\langle x, y_i \rangle\}\|_{X_d}$ are equivalent, and

(c) $x = \sum_{i=1}^{\infty} \langle x, y_i \rangle \, x_i$ for each $x \in X$,

then $(\{y_i\}, \{x_i\})$ is a (linear) *atomic decomposition* of $X$ with respect to $X_d$. If the norm equivalence is given by $A \, \|x\|_X \le \|\{\langle x, y_i \rangle\}\|_{X_d} \le B \, \|x\|_X$, then $A$, $B$ are a choice of *atomic bounds* for $(\{y_i\}, \{x_i\})$. $\square$

Some examples of atomic decompositions in Banach function spaces are the "$\phi$-transform" of Frazier and Jawerth [FJ] and Feichtinger and Gröchenig's constructions [FG], [G].

An atomic decomposition provides a factorization of the identity map $I$ on $X$. That is, $I$ is written as a composition of the *coefficient mapping* $x \mapsto \{\langle x, y_i \rangle\}$ and the *reconstruction operator* $\{c_i\} \mapsto \sum c_i \, x_i$. Such series reconstructions are theoretically appealing. However, for numerical implementations it is often preferable to formulate the reconstruction operator via an iteration or other algorithm. We therefore make the following definition, which allows freedom in the form of the reconstruction operator.

Definition 2. Let $X$ be a Banach space and let $X_d$ be an associated Banach space of scalar-valued sequences indexed by $\mathbf{N}$. Let $\{y_i\}_{i \in \mathbf{N}} \subset X'$ and $S\colon X_d \to X$ be given. If:

(a) $\{\langle x, y_i \rangle\} \in X_d$ for each $x \in X$,

(b) The norms $\|x\|_X$ and $\|\{\langle x, y_i \rangle\}\|_{X_d}$ are equivalent, and

(c) $S$ is bounded and linear, and $S\{\langle x, y_i \rangle\} = x$ for each $x \in X$,

then $(\{y_i\}, S)$ is a *Banach frame* for $X$ with respect to $X_d$. The mapping $S$ is the *reconstruction operator.* If the norm equivalence is given by $A \, \|x\|_X \le \|\{\langle x, y_i \rangle\}\|_{X_d} \le B \, \|x\|_X$, then $A$, $B$ are a choice of *frame bounds* for $(\{y_i\}, S)$. $\square$

Note that if $U\colon X \to X_d$ is the coefficient mapping defined by $Ux = \{\langle x, y_i \rangle\}$ then

$\|S\|^{-1}$, $\|U\|$ are a choice of frame bounds for the Banach frame $(\{y_i\}, S)$.

Our purpose in this note is to prove that atomic decompositions and Banach frames are stable under small perturbations. This is inspired by corresponding classical perturbation results, e.g., the Paley-Wiener basis stability criteria [PW], [Y] and the perturbation theorem of Kato [K]. We introduce new and weaker conditions which still ensure the desired stability. This is not only of theoretical interest: we show that our results can be applied to coherent state frames. This is not the case if standard basis perturbation results are merely generalized directly to frames. (Retherford and Holub [RH] is an excellent survey of perturbation results for bases.) In addition, we investigate duality properties of atomic decompositions, and consider the consequences of our results for Hilbert frames. Related Hilbert space results, partially inspired by a weak version of Theorem 2 first proved in [He], have appeared in [C1], [C2], [C3].

We assume throughout that sequences are indexed by $\mathbf{N}$. We use the following terminology. We say that a sequence $\{c_i\}$ is *finite* if only finitely many components are nonzero. A Banach space $X_d$ of sequences is *solid* if whenever $\{b_i\}$ and $\{c_i\}$ are sequences with $\{c_i\} \in X_d$ and $|b_i| \le |c_i|$ then it follows that $\{b_i\} \in X_d$ and $\|\{b_i\}\|_{X_d} \le \|\{c_i\}\|_{X_d}$. For example, let $e_j$ denote the delta sequence $e_j(i) = \delta_{ij}$. If $\{e_j\}_{j\in\mathbf{N}}$ forms an unconditional basis for $X_d$ then $X_d$ is solid. It also follows in this case that $X_d$ has an *absolutely continuous norm*, i.e., if $\{c_i\} \in X_d$ then $\lim_{n\to\infty} \|\{c_i - c_i \cdot \chi_{I_n}(i)\}\|_{X_d} = 0$, where $\{I_n\}$ is any family of subsets of $\mathbf{N}$ such that $I_1 \subset I_2 \subset \cdots \nearrow \mathbf{N}$ and $\chi_{I_n}$ is the characteristic function $\chi_{I_n}(i) = 1$ if $i \in I_n$, 0 if $i \notin I_n$. In particular, the hypotheses on $X_d$, $X_d'$ in most of our results are satisfied if $X_d'$ can be realized as a Banach space of sequences of scalars and if $\{e_j\}$ forms an unconditional basis for both $X_d$ and $X_d'$.

**2. Perturbation results.** We first show that Banach frames are stable under small perturbations of the frame elements.

THEOREM 1. *Let $(\{y_i\}, S)$ be a Banach frame for $X$ with respect to $X_d$. Let $\{z_i\} \subset X'$. If there exist $\lambda$, $\mu \ge 0$ such that*

(a) $\lambda\,\|U\| + \mu \;<\; \|S\|^{-1}$, *and*

(b) $\|\{\langle x, y_i - z_i\rangle\}\|_{X_d} \;\le\; \lambda\,\|\{\langle x, y_i\rangle\}\|_{X_d} \;+\; \mu\,\|x\|_X$ *for all* $x \in X$,

*then there exists a reconstruction operator $T$ such that $(\{z_i\}, T)$ is a Banach frame for $X$*

*with respect to* $X_d$ *with frame bounds* $\|S\|^{-1} - (\lambda\,\|U\| + \mu)$, $\|U\| + (\lambda\,\|U\| + \mu)$, *where* $U$ *is the coefficient mapping* $Ux = \{\langle x, y_i\rangle\}$.

*Proof.* The hypotheses imply that the operator $V: X \to X_d$ defined by $Vx = \{\langle x, z_i\rangle\}$ is bounded and satisfies

$$\|Ux - Vx\|_{X_d} \;\leq\; \lambda\,\|Ux\|_{X_d} \;+\; \mu\,\|x\|_X$$

for all $x \in X$. Therefore,

$$\|Vx\|_{X_d} \;\leq\; (\|U\| \;+\; \lambda\,\|U\| \;+\; \mu)\,\|x\|_X.$$

This establishes the upper frame bound. For the lower bound, observe that $SU = I$, so

$$\|I - SV\| \;\leq\; \|S\|\,\|U - V\| \;\leq\; \|S\|\,(\lambda\,\|U\| \;+\; \mu) \;<\; 1.$$

Therefore $SV$ is invertible, and $\|(SV)^{-1}\| \leq (1 - (\lambda\,\|U\| + \mu)\,\|S\|)^{-1}$. Finally, if we set $T = (SV)^{-1}S$ then $TV = I$, and

$$\|x\|_X \;\leq\; \|T\|\,\|Vx\|_{X_d} \;\leq\; \frac{\|S\|}{1 - (\lambda\,\|U\| + \mu)\,\|S\|}\;\|Vx\|_{X_d}.$$

This gives the desired lower bound:

$$\left(\frac{1}{\|S\|} \;-\; (\lambda\,\|U\| + \mu)\right)\,\|x\|_X \;\leq\; \|Vx\|_{X_d}. \qquad \square$$

The hypotheses in Theorem 1 are natural from the point of view of perturbation of operators: they mean that the operator $U - V$ is *relatively bounded* with respect to $U$ [K, p. 181]. In Section 4 we apply this result to Hilbert frames.

For atomic decompositions, we can perturb in $X$ instead of $X'$. Our result is a "Paley-Wiener Theorem for atomic decompositions" [Y, p. 38].

THEOREM 2. *Suppose that* $X_d$ *has an absolutely continuous norm. Let* $(\{y_i\}, \{x_i\})$ *be an atomic decomposition of* $X$ *with respect to* $X_d$ *with bounds* $A$, $B$. *Let* $\{w_i\} \subset X$. *If there exist* $\lambda$, $\mu \geq 0$ *such that*

(a) $\lambda + \mu\,B < 1$, *and*

(b) $\left\|\sum c_i\,(x_i - w_i)\right\|_X \;\leq\; \lambda\left\|\sum c_i\,x_i\right\|_X + \mu\,\|\{c_i\}\|_{X_d}$ *for any finite sequence* $\{c_i\} \in X_d$,

*then there exists a family* $\{z_i\} \subset X'$ *such that* $(\{z_i\}, \{w_i\})$ *is an atomic decomposition of* $X$ *with respect to* $X_d$ *with bounds* $A\,(1 + (\lambda + \mu B))^{-1}$, $B\,(1 - (\lambda + \mu B))^{-1}$. *Moreover,*

*$\{w_i\}$ is a basis for $X$ if and only if $\{x_i\}$ is a basis for $X$.*

*Proof.* Because of the assumption that $X_d$ has an absolutely continuous norm, the series $\sum \langle x, y_i\rangle\, w_i$ is convergent for any $x \in X$. If we define $T\colon X \to X$ by $Tx = \sum \langle x, y_i\rangle\, w_i$, then

$$\|x - Tx\|_X \;\le\; \lambda\, \|x\|_X \;+\; \mu\, \|\{\langle x, y_i\rangle\}\|_{X_d} \;\le\; (\lambda + \mu\, B)\, \|x\|_X$$

for all $x \in X$. Therefore $\|I - T\| < 1$, so $T$ is invertible. Define $z_i = (T^{-1})^* y_i$. Then

$$x \;=\; TT^{-1}x \;=\; \sum \langle T^{-1}x, y_i\rangle\, w_i \;=\; \sum \langle x, z_i\rangle\, w_i.$$

Further,

$$\frac{A}{\|T\|}\, \|x\|_X \;\le\; A\, \|T^{-1}x\|_X \;\le\; \|\{\langle T^{-1}x, y_i\rangle\}\|_{X_d} \;\le\; B\, \|T^{-1}x\|_X \;\le\; B\, \|T^{-1}\|\, \|x\|_X,$$

so $(\{z_i\}, \{w_i\})$ is an atomic decomposition of $X$ with respect to $X_d$. Since $\|T\| \le 1 + \lambda + \mu\, B$ and $\|T^{-1}\| \le (1 - (\lambda + \mu\, B))^{-1}$, the bounds are as claimed.

Finally, assume that $\{x_i\}$ is a basis for $X$. Then $\{x_i\}$ and $\{y_i\}$ are biorthonormal, so $Tx_j = \sum \langle T^{-1}Tx_j, y_i\rangle\, w_i = w_j$. Therefore $\{w_i\}$ is a basis since $T$ is invertible. Conversely, if $\{w_i\}$ is a basis then $T^{-1}$ maps it onto $\{x_i\}$. $\square$

In the terminology of Kato [K, p. 181], the hypotheses in Theorem 2 are that the operator $K\colon \mathcal{D}(K) \subset X_d \to X$ defined by $K\{c_i\} = \sum c_i\, (x_i - w_i)$ is relatively bounded with respect to the operator $\{c_i\} \mapsto \sum c_i\, x_i$. It is natural to call $K$ the "perturbation operator," since, as we have seen, conditions on $K$ imply that "$\{w_i\}$ inherits decomposition properties from $\{x_i\}$."

We point out some consequences of Theorem 2. First, specific choices of $\lambda$ and $\mu$ give conditions in the style of classic results on basis perturbation.

Corollary 3. *Let $(\{y_i\}, \{x_i\})$ be an atomic decomposition of $X$ with respect to $X_d$ with bounds $A$, $B$. Assume that $X_d$, $X_d'$ satisfy:*

(a) *$X_d$ has an absolutely continuous norm,*

(b) *$X_d'$ is a solid Banach space of scalar-valued sequences, and*

(c) *the action of $\{c_i\} \in X_d'$ on $\{b_i\} \in X_d$ is given by $\langle \{b_i\}, \{c_i\}\rangle = \sum b_i\, \bar{c}_i$.*

*If* $\{w_i\} \subset X$ *is such that*

$$R \;=\; \|\{\|x_i - w_i\|_X\}\|_{X_d'} \;<\; \frac{1}{B},$$

*then there exists a family* $\{z_i\} \subset X'$ *such that* $(\{z_i\}, \{w_i\})$ *is an atomic decomposition of* $X$ *with respect to* $X_d$ *with bounds* $A\,(1 + R\,B)^{-1}$, $B\,(1 - R\,B)^{-1}$.

*Proof.* The hypotheses imply that

$$\Bigl\|\sum_i c_i\,(x_i - w_i)\Bigr\|_X \;\le\; R\,\|\{c_i\}\|_{X_d}$$

for any finite sequence $\{c_i\} \in X_d$. Therefore we can apply Theorem 2 with $\lambda = 0$ and $\mu = R$. $\square$

A drawback of Corollary 3 is that it generally does not apply to the problem of perturbing the mother wavelet of a *coherent state* atomic decomposition. These are the most important practical incarnations of atomic decompositions. A coherent state atomic decomposition has $x_i = \pi(g_i)x$, where $\pi$ is a representation of a group $G$ on $X$ such that each $\pi(g)$ is a bijective isometry of $X$ onto itself, $\{g_i\}$ is a discrete set in $G$, and $x \in X$ is the *generator* or, by an abuse of terminology, the *mother wavelet*. (See [HW] for examples of typical groups and representations). When the mother wavelet $x$ is perturbed, say to $w$, we have $\|\pi(g_i)x - \pi(g_i)w\|_X \equiv \|x - w\|_X$. Hence $\|\{\|\pi(g_i)x - \pi(g_i)w\|_X\}\|_{X_d'}$ will typically be infinite. On the other hand, the hypotheses of Theorem 2 can still be applicable (we discuss this further in the Hilbert space setting following Corollary 6). Feichtinger and Gröchenig [FG] have proved some perturbation results for coherent state atomic decompositions. The novelty of Theorem 2 is its general formulation and proof.

We say that a sequence $\{x_i\} \subset X$ is a *Bessel sequence* for $X'$ with respect to $X_d'$ if there exists a constant $D$ such that

$$\|\{\langle x_i, y\rangle\}\|_{X_d'} \;\le\; D\,\|y\|_{X'} \quad \text{for all } y \in X'.$$

The constant $D$ is the *Bessel bound*. The following additional consequence of Theorem 2 is motivated by a useful result about Riesz bases in Hilbert spaces [Hi].

Corollary 4. *Let* $(\{y_i\}, \{x_i\})$ *be an atomic decomposition of* $X$ *with respect to* $X_d$ *with bounds* $A$, $B$, *and such that* $\{x_i\}$ *is a Bessel sequence for* $X'$ *with respect to* $X_d'$ *with*

*Bessel bound $D$. Assume that $X_d$, $X_d'$ satisfy hypotheses* (a), (b), *and* (c) *of Corollary* 3. *Assume that there exists a family $\{T_k\}$ of bounded operators on $X$ and scalars $a_{ik}$ so that*

$$x_i - w_i \;=\; \sum_k a_{ik}\, T_k x_i \quad \text{for each } i.$$

*If*

(a) $a_k = \sup_i |a_{ik}| < \infty$ *for each $k$, and*

(b) $\sum a_k \, \|T_k\| < (BD)^{-1}$,

*then there exists a family $\{z_i\} \subset X'$ such that $(\{z_i\}, \{w_i\})$ is an atomic decomposition of $X$ with respect to $X_d$ with bounds $A\,(1 + BD \sum a_k \, \|T_k\|)^{-1}$, $B\,(1 - BD \sum a_k \, \|T_k\|)^{-1}$.*

*Proof.* Given a finite sequence $\{c_i\} \in X_d$, we have

$$\Big\| \sum_i c_i \,(x_i - w_i) \Big\|_X \;=\; \Big\| \sum_i c_i \, \sum_k a_{ik}\, T_k x_i \Big\|_X \;\le\; \sum_k \|T_k\| \, \Big\| \sum_i c_i \, a_{ik}\, x_i \Big\|_X .$$

Fix any $k$. Then

$$\begin{aligned} \Big\| \sum_i c_i \, a_{ik}\, x_i \Big\|_X &= \sup_{\|y\|_{X'}=1} \Big| \sum_i c_i \, a_{ik}\, \langle x_i, y\rangle \Big| \\ &= \sup_{\|y\|_{X'}=1} \big| \langle \{c_i\}, \{\overline{a_{ik}\,\langle x_i, y\rangle}\}\rangle \big| \\ &\le \|\{c_i\}\|_{X_d} \sup_{\|y\|_{X'}=1} \|\{a_{ik}\,\langle x_i, y\rangle\}\|_{X_d'} \\ &\le D\, a_k \, \|\{c_i\}\|_{X_d}, \end{aligned}$$

where we have used the fact that $X_d'$ is solid. Hence

$$\Big\| \sum_i c_i \,(x_i - w_i) \Big\|_X \;\le\; D \left( \sum_k a_k \, \|T_k\| \right) \|\{c_i\}\|_{X_d}$$

for every finite sequence $\{c_i\} \in X_d$. We can therefore apply Theorem 2 with $\lambda = 0$ and $\mu = D \sum a_k \, \|T_k\|$. □

**3. Duality for atomic decompositions.** If $\{x_i\}$ is a basis for $X$ with coefficient functionals $\{y_i\}$ then $\{y_i\}$ is a basis for $\overline{\mathrm{span}}\{y_i\} \subset X'$ with coefficient functions $\{x_i\} \subset X''$. We investigate the analogous question for atomic decompositions.

THEOREM 5. *Let* $(\{y_i\}, \{x_i\})$ *be an atomic decomposition of* $X$ *with respect to* $X_d$. *Assume* $X_d$, $X'_d$ *satisfy:*

(a) $X_d$ *is solid,*

(b) $X'_d$ *is a Banach space of scalar-valued sequences,*

(c) *the action of* $\{c_i\} \in X'_d$ *on* $\{b_i\} \in X_d$ *is given by* $\langle \{b_i\}, \{c_i\} \rangle = \sum b_i \, \bar{c}_i$, *and*

(d) $X'_d$ *has an absolutely continuous norm.*

*If* $\{x_i\}$ *is a Bessel sequence for* $X'$ *with respect to* $X'_d$ *then* $(\{x_i\}, \{y_i\})$ *is an atomic decomposition of* $X'$ *with respect to* $X'_d$.

*Proof.* The hypotheses given imply that $\sum c_i \, y_i$ converges in $X'$ for every $\{c_i\} \in X'_d$. In particular, since $\{x_i\}$ is a Bessel sequence, if $y \in X'$ is fixed then $\{\langle x_i, y \rangle\} \in X'_d$, so $\sum \langle x_i, y \rangle \, y_i$ converges in $X'$. Moreover, if $x \in X$ then

$$\Big\langle x, \ \sum \langle x_i, y \rangle \, y_i \Big\rangle \ = \ \Big\langle \sum \langle x, y_i \rangle \, x_i, \ y \Big\rangle \ = \ \langle x, y \rangle.$$

Hence $\sum \langle x_i, y \rangle \, y_i = y$ for each $y \in X'$.

It remains only to show that there is a constant $C$ such that $C \, \|y\|_{X'} \le \|\{\langle x_i, y \rangle\}\|_{X'_d}$ for all $y \in X'$. However, if $y \in X'$ then

$$\begin{aligned}
\|y\|_{X'} &= \sup_{\|x\|_X = 1} |\langle x, y \rangle| \\
&= \sup_{\|x\|_X = 1} \Big| \sum \langle x, y_i \rangle \, \langle x_i, y \rangle \Big| \\
&\le \sup_{\|x\|_X = 1} \|\{\langle x, y_i \rangle\}\|_{X_d} \, \|\{\langle x_i, y \rangle\}\|_{X'_d} \\
&\le B \, \|\{\langle x_i, y \rangle\}\|_{X'_d},
\end{aligned}$$

so the proof is complete. $\square$

The Bessel sequence hypothesis is clearly necessary. For example, suppose $(\{y_i\}, \{x_i\})$ is an atomic decomposition of $X$ with respect to $X_d$ and that $\{w_j\}$ is not a Bessel sequence for $X'$ with respect to $X'_d$. Define $z_j = 0$ for each $j$; then $(\{y_i\} \cup \{z_j\}, \{x_i\} \cup \{w_j\})$ is an atomic decomposition of $X$ with respect to $X_d$, although $(\{x_i\} \cup \{w_j\}, \{y_i\} \cup \{z_j\})$ is not an atomic decomposition of $X'$ with respect to $X'_d$.

**4. Frame decompositions in Hilbert spaces.** In this section we consider the case $X = H$, a separable Hilbert space, and $X_d = \ell^2$. For Hilbert frames, it is customary to use a definition of frame bounds slightly different from the one we gave for Banach frames in Definition 2. In particular, if $\{x_i\}$ is a Hilbert frame then the norm equivalence between $\|x\|_H$ and $\|\{\langle x, x_i\rangle\}\|_{\ell^2}$ is usually written

$$A\,\|x\|_H^2 \;\le\; \sum_i |\langle x, x_i\rangle|^2 \;\le\; B\,\|x\|_H^2 \quad \text{for all } x \in H, \tag{2}$$

with these $A$, $B$ called the frame bounds. For clarity, we will refer to $A$, $B$ given by (2) as *Hilbert frame bounds*; they are the squares of the Banach frame bounds given in Definition 2.

First we prove an important consequence of Theorem 1.

Corollary 6. *Let $\{x_i\}$ be a Hilbert frame with Hilbert frame bounds $A$, $B$. Let $\{w_i\} \subset H$. If there is an $R < A$ such that*

$$\sum_i |\langle x, x_i - w_i\rangle|^2 \;\le\; R\,\|x\|_H^2 \quad \textit{for all } x \in H, \tag{3}$$

*then $\{w_i\}$ is a Hilbert frame with Hilbert frame bounds $A\,(1-\sqrt{R/A})^2$, $B\,(1+\sqrt{R/B})^2$.*

*Proof.* Let $\{\tilde{x}_i\}$ be the dual frame of $\{x_i\}$. If we define $S\{c_i\} = \sum c_i\,\tilde{x}_i$, then $(\{x_i\}, S)$ is a Banach frame for $H$ with respect to $\ell^2$ with Banach frame bounds $\sqrt{A}$, $\sqrt{B}$. By standard Hilbert space arguments [DS],

$$\Big\|\sum_i c_i\,\tilde{x}_i\Big\|_H \;\le\; \frac{1}{\sqrt{A}}\,\|\{c_i\}\|_{\ell^2}$$

for every sequence $\{c_i\} \in \ell^2$. Therefore, we can apply Theorem 1 with $\lambda = 0$ and $\mu = \sqrt{R}$ to obtain

$$(\sqrt{A} - \sqrt{R})\,\|x\|_H \;\le\; \Big(\sum_i |\langle x, w_i\rangle|^2\Big)^{1/2} \;\le\; (\sqrt{B} + \sqrt{R})\,\|x\|_H. \qquad \square$$

In most cases it is more difficult to verify the lower frame condition than the upper one. Corollary 6 shows that "the difficult problem reduces to the easier one in the case of perturbation": the family $\{w_i\}$ is a frame if the difference $\{x_i - w_i\}$ satisfies the upper condition with a sufficiently small bound. Note that this is a weaker hypothesis than the standard basis-type assumption that $\sum \|x_i - w_i\|_H^2 < A$. In particular, this latter hypothesis cannot be applied to the problem of perturbing the mother wavelet $x$ of a coherent state frame $\{\pi(g_i)x\}$. However, Corollary 6 does apply to this problem: it states that $\{\pi(g_i)w\}$ is a frame if the set of coherent states $\{\pi(g_i)(x-w)\}$ generated by $x-w$ is a Bessel sequence with bound less than $A$. As noted above, establishing that $\{\pi(g_i)(x-w)\}$ is a Bessel sequence is usually not a difficult matter. For example, Favier and Zalik [FZ] obtain such results explicitly for the case of Gabor frames (frames where $\pi$ is the Schroedinger representation of the Heisenberg group on $L^2(\mathbf{R})$). For applications of Corollary 6 to other problems in irregular sampling and wavelet theory, we refer to [FZ] and [C3].

We have already remarked on the importance of the perturbation operator $K$. For Hilbert frames we are able to prove another result where $K$ plays the main role.

Theorem 7. *Let $\{x_i\}$ be a Hilbert frame for $H$, and let $\{w_i\} \subset H$. If $K\{c_i\} = \sum c_i (w_i - x_i)$ is compact as an operator from $\ell^2$ into $H$, then $\{w_i\}$ is a Hilbert frame for $\overline{\mathrm{span}}\{w_i\}$.*

*Proof.* Define $T: \ell^2 \to H$ by $T\{c_i\} = \sum c_i x_i$. Since $\{x_i\}$ is a frame, we know that $T$ is bounded. In fact, $\|T\|^2 \le B$, the upper Hilbert frame bound for $\{x_i\}$. Hence $V = T + K$ is a bounded operator from $\ell^2$ into $H$. If $x \in H$ then we compute

$$\sum |\langle x, w_i\rangle|^2 \;=\; \|V^*x\|_H^2 \;\le\; \|T+K\|^2\,\|x\|_H^2 \;\le\; B\left(1 + \frac{\|K\|}{\sqrt{B}}\right)\|x\|_H^2.$$

This establishes that $\{w_i\}$ satisfies an upper frame bound. The hypothesis that $K$ is compact will give us the existence of the lower frame bound, but it will not give a concrete value.

By [C1, Theorem 2.1], to show the existence of the lower frame bound for $\{w_i\}$, it suffices to show that the "frame operator" $VV^*$ for $\{w_i\}$ is surjective. Now,

$$VV^* \;=\; S + TK^* + KT^* + KK^*,$$

where $S = TT^*$ is the frame operator for $\{x_i\}$. The operator

$$(TK^* + KT^* + KK^*)\,S^{-1}$$

is compact, so the operator $(TK^* + KT^* + KK^*)\,S^{-1} + I$ has closed range [R, Theorem 4.23]. Composing this with $S$, we see that $VV^*$ also has closed range.

Now consider $VV^*$ as an operator on the closed subspace $\overline{\text{span}}\{w_i\}$. Here $VV^*$ is injective: if $x \in \overline{\text{span}}\{w_i\}$ and $VV^*x = 0$ then $\sum |\langle x, w_i\rangle|^2 = \langle VV^*x, x\rangle = 0$, whence $x = 0$. Since $VV^*$ has a closed range we therefore have $\text{Range}(VV^*) = (N(VV^*))^- = \overline{\text{span}}\{w_i\}$. Thus $VV^*$ is surjective, as desired, and hence $\{w_i\}$ is a frame for $\overline{\text{span}}\{w_i\}$. ☐

In particular, $\{w_i\}$ is a frame for $\overline{\text{span}}\{w_i\}$ if $\sum \|x_i - w_i\|_H^2 < \infty$. By Corollary 6 we know that if $\sum \|x_i - w_i\|_H^2 < A$ (the lower Hilbert frame bound for $\{x_i\}$) then $\{w_i\}$ is a frame for $H$, and therefore $\overline{\text{span}}\{w_i\} = H$. However, if we have merely the equality $\sum \|x_i - w_i\|_H^2 = A$, it may happen that $\overline{\text{span}}\{w_i\} \neq H$. For example, let $\{x_i\}$ be an orthonormal basis for $H$, and set $w_1 = 0$, $w_i = x_i$ for $i > 1$.

Also, note that the condition (3) in Corollary 6 is precisely the statement that $\|K\| < \sqrt{A}$. If $\|K\| \geq \sqrt{A}$ then $\{w_i\}$ need not be a frame for $\overline{\text{span}}\{w_i\}$. For example, if $\{x_i\}$ is an orthonormal basis for $H$ and we set $w_i = x_i + x_{i+1}$, then $\|K\| = A = 1$ but $\{w_i\}$ is not a frame for $\overline{\text{span}}\{w_i\} = H$.

Our final result establishes the relation between Hilbert frames and atomic decompositions in Hilbert spaces. Note that if $\{y_i\}$ is a Hilbert frame for $H$ then $(\{y_i\}, \{\tilde{y}_i\})$ is an atomic decomposition of $H$ with respect to $\ell^2$, where $\{\tilde{y}_i\}$ is the dual frame of $\{y_i\}$. The converse requires additional hypotheses.

THEOREM 8. *Let $(\{y_i\}, \{x_i\})$ be an atomic decomposition of $H$ with respect to $\ell^2$. Then the following statements hold.*

(a) *$\{y_i\}$ is a Hilbert frame for $H$.*

(b) *If $\{x_i\}$ is a Bessel sequence for $H$ with respect to $\ell^2$ then it is a Hilbert frame for $H$.*

(c) *Assume $\{x_i\}$ is a Bessel sequence for $H$ with respect to $\ell^2$. Define $U, V \colon H \to \ell^2$ by $Ux = \{\langle x, x_i\rangle\}$ and $Vx = \{\langle x, y_i\rangle\}$. Then $\{x_i\}$ is the dual frame of $\{y_i\}$ if and only if* $\text{Range}(U) = \text{Range}(V)$.

*Proof.* Statement (a) follows immediately from the definition. For (b), the lower frame bound follows from Theorem 5, or directly from the computation

$$\|x\|_H^4 = \left(\sum_i \langle x, y_i\rangle \langle x_i, x\rangle\right)^2 \leq \sum_i |\langle x, y_i\rangle|^2 \sum_i |\langle x_i, x\rangle|^2 \leq B^2 \|x\|_H^2 \sum_i |\langle x_i, x\rangle|^2.$$

Finally, for (c), note that the reconstruction formula (1) implies $U^*V = V^*U = I$.

Let $E = \mathrm{Range}(U)$. Since $U$ is injective and $UV^*U = U$, we have $(UV^*)|_E = I|_E$. If $E = \mathrm{Range}(V)$, this implies $UV^*V = V$. Therefore, given $x \in H$,

$$\{\langle x, y_i\rangle\} = Vx = UV^*Vx = \{\langle V^*Vx, x_i\rangle\} = \{\langle x, V^*Vx_i\rangle\}.$$

In particular, we must have $y_i = V^*Vx_i$, whence $x_i = (V^*V)^{-1}y_i$ and $\{x_i\}$ is the dual frame of $\{y_i\}$. Conversely, if $\{x_i\}$ is the dual frame of $\{y_i\}$ then $Vx = UV^*Vx$, so $\mathrm{Range}(V) = \mathrm{Range}(U)$ since $V^*V$ is invertible. $\square$

It need not be the case that $\{x_i\}$ is the dual frame of $\{y_i\}$ even if $(\{y_i\}, \{x_i\})$ is an atomic decomposition of $H$ with respect to $\ell^2$ and $\{x_i\}$ is a Bessel sequence. For example, let $\{y_i\}$ and $\{z_j\}$ be two frames for $H$. Define $w_j = 0$ for each $j$. Then $(\{y_i\} \cup \{z_j\}, \{\tilde{y}_i\} \cup \{w_j\})$ is an atomic decomposition of $H$ with respect to $\ell^2$, but the dual frame of $\{y_i\} \cup \{z_j\}$ is $\{\tilde{y}_i\} \cup \{\tilde{z}_j\}$.

We close with a note about convergence. The hypotheses on $X_d$, $X_d'$ used in most of the results were needed to ensure that series such as $\sum c_i\, y_i$ converge unconditionally for every $\{c_i\}$ in the appropriate sequence space. In the Hilbert setting, we know that if $\{y_i\}$ is a Hilbert frame then $\sum c_i\, y_i$ converges unconditionally in $H$ for every $\{c_i\} \in \ell^2$. In fact, this is true if $\{y_i\}$ is merely a Bessel sequence for $H$ with respect to $\ell^2$.

Moreover, if $\{y_i\}$ is an arbitrary sequence in $H$ and $\sum c_i\, y_i$ converges unconditionally, then Orlicz' Theorem implies $\sum |c_i|^2\, \|y_i\|_H^2 = \sum \|c_i\, y_i\|_H^2 < \infty$. Therefore, if $\{y_i\}$ is norm-bounded below (meaning $\inf \|y_i\|_H > 0$), then $\sum |c_i|^2 < \infty$.

In particular, if $\{y_i\}$ is a Bessel sequence for $H$ with respect to $\ell^2$ and $\{y_i\}$ is norm-bounded below, then

$$\{c_i\} \in \ell^2 \quad \Longleftrightarrow \quad \sum c_i\, y_i \text{ converges unconditionally in } H.$$

It would be useful to similarly characterize unconditional convergence in the Banach space setting.

**Acknowledgments.** We thank Fred Andrew, John Benedetto, Hans Feichtinger, Karlheinz Gröchenig, Henrik Stetkær and David Walnut for valuable discussions and insights.

## REFERENCES

[C1] O. Christensen, *Frames and the projection method*, Appl. Comp. Harm. Anal., 1 (1993), pp. 50–53.

[C2] ———, *A Paley–Wiener Theorem for Frames*, Proc. Amer. Math. Soc., to appear.

[C3] ———, *Moment problems and stability results for frames*, SIAM J. Math. Anal. (submitted).

[D1] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory, 39 (1990), pp. 961–1005.

[D2] ———, *Ten Lectures on Wavelets*, SIAM Press, Philadelphia, 1992.

[DGM] I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., 27 (1986), pp. 1271–1283.

[DS] R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., 72 (1952), pp. 341–366.

[FJ] M. Frazier and B. Jawerth, *Decompositions of Besov spaces*, Indiana Univ. Math. J., 34 (1985), pp. 777–799.

[FG] H.G. Feichtinger and K. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions,* I, J. Funct. Anal., 86 (1989), pp. 307–340; *Banach spaces related to integrable group representations and their atomic decompositions,* II, Monatshefte für Mathematik, 108 (1989), pp. 129–148.

[FZ] S.J. Favier and R.A. Zalik, *On the stability of frames*, preprint, June 1994.

[G] K. Gröchenig, *Describing functions: Atomic decompositions versus frames*, Monatshefte für Mathematik, 112 (1991), pp. 1–41.

[He] C. Heil, *Wiener amalgam spaces in generalized harmonic analysis and wavelet theory*, Ph.D. Thesis, University of Maryland, College Park, MD, 1990.

[Hi] J.R. Higgins, *Completeness and Basis Properties of Sets of Special Functions*, Cambridge University Press, 1976.

[HW] C. Heil and D. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review, 31 (1989), pp. 628–666.

[K] T. Kato, *Perturbation Theory for Linear Operators, Second Edition*, Springer–Verlag, New York, 1976.

[PW] R.E.A.C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications Vol. 19, American Mathematical Society, New York, 1934.

[RH] J.R. Retherford and J.R. Holub, *The stability of bases in Banach and Hilbert spaces*, J. Reine Angew. Math., 246 (1971), pp. 136–146.

[R] W. Rudin, *Functional Analysis, Second Edition*, McGraw–Hill, New York, 1991.

[Y] R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.